Bäcklund transformation and special solutions for Drinfeld–Sokolov–Satsuma–Hirota system of coupled equations

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Abstract

Using the Weiss method of truncated singular expansions, we construct an explicit Bäcklund transformation of the Drinfeld–Sokolov–Satsuma–Hirota system into itself. Then we find all the special solutions generated by this transformation from the trivial zero solution of this system.

Short title: *Bäcklund transformation and special solutions*
PACS numbers: *02.30.Ik, 02.30.Jr*

The system of two coupled nonlinear evolution equations

$$u_t + u_{xxx} - 6uu_x - 6v_x = 0 \qquad v_t - 2v_{xxx} + 6uv_x = 0 \tag{1}$$

was introduced, independently, by Drinfeld and Sokolov [1], and by Satsuma and Hirota [2]. In [1], the system (1) was given as one of numerous examples of nonlinear equations possessing Lax pairs of a special form. In [2], the system (1) was found as a special case of the four-reduction of the KP hierarchy, and its explicit one-soliton solution was constructed. Recently, Gürses and Karasu [3] found a recursion operator and a bi-Hamiltonian structure

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for (1), which provide the system (1) with an infinite algebra of generalized symmetries and an infinite set of conservation laws.

In this paper, we construct, using the Weiss method of truncated singular expansions [4], an explicit Bäcklund transformation of the Drinfeld–Sokolov–Satsuma–Hirota (DSSH) system (1) into itself, and then find all the special solutions generated by this transformation from the trivial zero solution of this system. In relation with our results, we should mention the recent paper of Tian and Gao [5], who did not find a Bäcklund transformation of (1) into itself, contrary to their claim, and produced a rational special solution of the DSSH system with erroneously many arbitrary parameters.

First of all, we find it useful to rewrite the DSSH system (1) in the form of the single sixth-order equation

$$w_{tt} - w_{xxxt} - 2w_{xxxxxx} + 18w_x w_{xxxx} + 36w_{xx}w_{xxx} - 36w_x^2 w_{xx} = 0 \qquad (2)$$

which is related to (1) by the Miura-type transformation

$$u = w_x \qquad v = \frac{1}{6}\left(w_t + w_{xxx} - 3w_x^2\right). \qquad (3)$$

This representation (2) of the original system (1) is very convenient, because the Bäcklund transformation of (1) would contain complicated radicals, whereas the Bäcklund transformation of (2) turns out to involve rational expressions only.

It is easy to verify that the equation (2) possesses the Painlevé property in the formulation for partial differential equations [6]. Substituting the singular expansion $w = w_0\phi^\alpha + \cdots + w_n\phi^{n+\alpha} + \cdots$ into (2), we find the following two branches, i.e. admissible choices of α and w_0 with corresponding positions n of resonances: (i) $\alpha = -1$, $w_0 = -2\phi_x$, $n = -1, 1, 3, 4, 6, 8$, and (ii) $\alpha = -1$, $w_0 = -10\phi_x$, $n = -5, -1, 1, 6, 8, 12$. Then, checking the consistency of the recursion relations for w_n at the resonances of both branches, we find that no logarithmic terms should be introduced into the singular expansions of solutions.

According to the positions of resonances, the branch (i) is generic, but the branch (ii) is not. Since the Bäcklund transformation sought should be applicable to a generic solution of the equation (2), we have to consider the truncated singular expansion in the branch (i). The use of the new expansion function χ, $\chi = \left(\phi^{-1}\phi_x - \frac{1}{2}\phi_x^{-1}\phi_{xx}\right)^{-1}$, proposed by Conte [7], simplifies the computations very considerably. Following [7], we substitute the truncated expansion $w = -2\chi^{-1} + a(x,t)$ into the equation (2), use the identities $\chi_x = 1 + \frac{1}{2}S\chi^2$, $\chi_t = -C + C_x\chi - \frac{1}{2}(C_{xx} + CS)\chi^2$ and $S_t + C_{xxx} + 2SC_x + CS_x = 0$, where $S = \phi_x^{-1}\phi_{xxx} - \frac{3}{2}\phi_x^{-2}\phi_{xx}^2$ and $C = -\phi_x^{-1}\phi_t$, collect terms with equal

degrees of χ, and thus obtain a complicated system of equations, which turns out to be compatible and equivalent to the following normal system of two equations for ϕ and a of total order six:

$$a_x + \frac{1}{6}C + \frac{1}{3}S = 0 \qquad a_t + a_{xxx} - \frac{3}{2}a_x^2 + \frac{3}{2}a_x C + \frac{3}{8}C^2 = k \tag{4}$$

where the parameter k appeared as a constant of integration.

The truncated singular expansion

$$w = -2\frac{\phi_x}{\phi} + \frac{\phi_{xx}}{\phi_x} + a \tag{5}$$

represents a Miura-type transformation between the system (4) and the equation (2). According to the Weiss method [4], the function $z\,(x,t)$ determined by

$$z = \frac{\phi_{xx}}{\phi_x} + a \tag{6}$$

is also a solution of (2). Therefore (6) represents one more Miura-type transformation between (4) and (2). These two Miura-type transformations, (5) and (6), together with the auxiliary system (4), determine an implicit Bäcklund transformation between the 'new' solution w and the 'old' solution z of the DSSH system written in the form (2) (certainly, these terms 'new' and 'old' can be used in the opposite order).

Now, eliminating ϕ and a from the equations (4), (5) and (6), we obtain the following explicit Bäcklund transformation of the equation (2) into itself:

$$p_t + \left(4p_{xx} - 3\frac{p_x^2}{p} - 3pq_x + \frac{1}{4}p^3\right)_x = 0 \tag{7}$$

$$\begin{aligned} q_t - \frac{3}{2}\frac{p_{xt}}{p} + \frac{3}{4}\frac{p_x p_t}{p^2} - \frac{1}{2}q_{xxx} - \frac{3}{2}\frac{q_x p_{xx}}{p} + \frac{3}{4}\frac{p_x q_{xx}}{p} + \frac{9}{8}pp_{xx} \\ + \frac{3}{4}\frac{p_x^2 q_x}{p^2} - \frac{9}{16}p_x^2 + \frac{3}{2}q_x^2 - \frac{3}{4}p^2 q_x + \frac{3}{64}p^4 = 2k \end{aligned} \tag{8}$$

where $p = w - z$ and $q = w + z$. The equations (7) and (8) constitute a Bäcklund transformation in the sense of the definition given in [8]: if we eliminate z from (7) and (8), we obtain exactly the equation (2) for w; and if we eliminate w from (7) and (8), we obtain (2) for z. We think, however, that it is impossible to prove this by hand-made computations: we did it by means of the *Mathematica* system [9]. As far as we know, the equations (7)

and (8) are the most complicated explicit Bäcklund transformation in the literature.

Let us find all the 'new' solutions w of the equation (2) generated by the obtained Bäcklund transformation from the trivial 'old' solution $z = 0$. Setting $z = 0$ in (7) and (8), we get an over-determined system of two equations for w, which can be solved exactly, using the new dependent variable ψ: $w = -2\phi^{-1}\phi_x$, $\phi_x = \psi^2$. This gives us the following five types of explicit special solutions of the DSSH system written in the form (2):

$$w = \frac{-2}{x} \tag{9}$$

$$w = \frac{-6x^2}{x^3 + 12t} \tag{10}$$

$$w = \frac{-30\left(x^2 + \sigma\right)^2}{3x^5 + 10\sigma x^3 + 15\sigma^2 x - 240\sigma t + \tau} \tag{11}$$

$$\begin{aligned} w = &-210\left(x^3 + \sigma x - 24t\right)^2 \left(15x^7 + 42\sigma x^5 - 1260x^4 t\right. \\ &\left.+35\sigma^2 x^3 - 2520\sigma x^2 t + 60480xt^2 + 420\sigma^2 t + \tau\right)^{-1} \end{aligned} \tag{12}$$

$$\begin{aligned} w = &-4\kappa\left(c_1 \mathrm{e}^{a} + c_2 \mathrm{e}^{-a} + c_3 \mathrm{e}^{b} + c_4 \mathrm{e}^{-b}\right)^2 \\ &\times\left[c_1^2 \mathrm{e}^{2a} - c_2^2 \mathrm{e}^{-2a} - \mathrm{i} c_3^2 \mathrm{e}^{2b} + \mathrm{i} c_4^2 \mathrm{e}^{-2b}\right. \\ &+2\left(1 - \mathrm{i}\right) c_1 c_3 \mathrm{e}^{a+b} + 2\left(1 + \mathrm{i}\right) c_1 c_4 \mathrm{e}^{a-b} \\ &-2\left(1 + \mathrm{i}\right) c_2 c_3 \mathrm{e}^{-a+b} - 2\left(1 - \mathrm{i}\right) c_2 c_4 \mathrm{e}^{-a-b} \\ &\left.+4\left(c_1 c_2 + c_3 c_4\right)\kappa x - 48\left(c_1 c_2 - c_3 c_4\right)\kappa^3 t + c_5\right]^{-1} \end{aligned} \tag{13}$$

where some inessential parameters were eliminated using arbitrary shifts $x \to x + x_0$ and $t \to t + t_0$; $\sigma, \tau, c_1, c_2, c_3, c_4, c_5$ and κ are arbitrary constants; $a = \kappa x - 4\kappa^3 t$ and $b = \mathrm{i}\kappa x + 4\mathrm{i}\kappa^3 t$; κ is related to k as $k = 6\kappa^4$; and the solutions (9)–(12) correspond to the case $k = 0$.

With respect to the DSSH system (1), the solutions (9) and (10) are trivial in the sense that, according to (3), $v = 0$ for them; actually, they are rational solutions of the potential KdV equation $w_t + w_{xxx} - 3w_x^2 = 0$. But the rational solutions (11) and (12) are not trivial in this sense: $v \neq 0$ for them. As for the solution (13), $v = 0$ only if $c_1 = c_2 = 0$ or $c_3 = c_4 = 0$. Consequently, the obtained Bäcklund transformation can produce nontrivial solutions for the coupled equations (1).

We should notice, however, that there are no solitary wave solutions among the solutions (9)–(13). Indeed, the only solution of the form $w = f(x - ct)$ ($c =$ constant), generated by (7)–(8) from $z = 0$, is the solution (9), for which $c = 0$. For this reason, we suspect (7)–(8) of being not the simplest (elementary) Bäcklund transformation of the equation (2) into itself: it might be a product of two elementary Bäcklund transformations; such a phenomenon was observed in [10], where two different Bäcklund transformations of the Calogero equation were found and studied. This point requires further investigation.

The authors are grateful to the Scientific and Technical Research Council of Turkey (TÜBİTAK) for support.

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